Suppression of Hamiltonian chaos by Coulomb repulsion in finite-amplitude electroconvection

R. Chacón
Departamento de Física Aplicada, Universidad de Castilla–La Mancha, 13004 Ciudad Real, Spain

A. T. Pérez* and A. Castellanos
Departamento de Electrónica y Electromagnetismo, Facultad de Física, Avenida Reina Mercedes s/n, 41012 Sevilla, Spain
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The role of Coulomb repulsion in finite-amplitude electroconvection is examined in relation to the existence of Hamiltonian chaos. It is shown that this Hamiltonian chaos is suppresed for high enough values of charge density.

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I. INTRODUCTION

There has been a long interest in electroconvection due to unipolar injection because of its theoretical and practical importance [1–3]. Experimental studies have shown that small fluctuations of the electrical current around its mean value are always present. The power spectra of electrical current indicate the existence of temporal chaos [4,5].

A theoretical attempt to explain this behavior was made in [6]. In that paper, the effect of a time-dependent velocity on the charge carriers trajectories in the case of very weak injection was studied. The ion trajectories correspond to the orbits of a Hamiltonian system and Melnikov's method was used to demonstrate the existence of Hamiltonian chaos.

In this Brief Report we study the case when injection is not weak enough to neglect Coulomb repulsion.

II. FORMULATION OF THE PROBLEM

Consider a dielectric liquid of permittivity ϵ, confined between two conducting parallel plates a distance d apart with an applied voltage difference Φ_0 between them. Injection takes place from one of the plates where a charge density q_0 is assumed to exist. When equations are put in nondimensional form taking as units d for distances, Φ_0/d for the electric field, $\epsilon\Phi_0/d^2$ for the charge density, and $K\Phi_0/d$ for the velocity (K is the ion mobility), the charge conservation equation is written

$$\frac{\partial q}{\partial t} + (\boldsymbol{u}+\boldsymbol{E})\cdot\boldsymbol{\nabla} q + q^2 = 0, \tag{1}$$

where $\boldsymbol{u}$ is the liquid velocity, $\boldsymbol{E}$ the electric field, and q the charge density. The electric field is related to the charge density through Poisson's equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = q. \tag{2}$$

The boundary conditions for these two equations are $\int_0^1 \boldsymbol{E}\cdot d\boldsymbol{r} = 1$ and $q = C$ at $z=0$. The coordinate system is chosen in such a way that the injecting electrode is at $z=0$ and the collector at $z=1$.

The parameter $C = q_0 d^2/\epsilon\Phi_0$ is a measure of the injected charge. The term q^2 in Eq. (1) represents the decrease of charge in a parcel of fluid due to the Coulomb repulsion between ions. When injection is very weak ($C \ll 1$) this term is negligible.

We assume that the fluid motion is two dimensional and in the form of a self-similar roll: $\boldsymbol{u} = A\boldsymbol{u}_0$, with $\max|\boldsymbol{u}_0| = 1$. We also introduce the stream function $\Psi_0(x,z) = (L/2\pi)(1-\cos 2\pi z)\sin \pi x/L$, such that

$$u_0 = \frac{\partial \Psi_0}{\partial z}, \tag{3a}$$

$$w_0 = -\frac{\partial \Psi_0}{\partial x}. \tag{3b}$$

Here L is the half length of a convective cell, which we assume to be 0.66 [3].

Equation (1) is equivalent to the following set of ordinary differential equation:

$$\frac{dq}{dt} = -q^2, \tag{4a}$$

$$\frac{dx}{dt} = Au_0(x,z) + E_x, \tag{4b}$$

$$\frac{dz}{dt} = Aw_0(x,z) + E_z. \tag{4c}$$

When Coulomb repulsion is neglected, Eq. (4) can be written in the form of Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial z}, \tag{5a}$$

$$\frac{dz}{dt} = -\frac{\partial H}{\partial x}, \tag{5b}$$

*Author to whom all correspondence should be addressed.

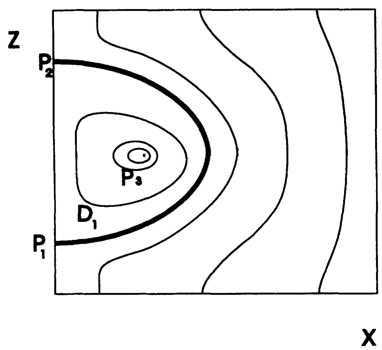

FIG. 1. Equilibria (P_1, P_2, and P_3), separatrix, and typical trajectories for a non-time-dependent amplitude.

where $H(x,z) = -x + A\Psi_0(x,z)$.

Typical trajectories are shown in Fig. 1 for a constant amplitude $A > 1$. The system has three equilibrium points P_1, P_2, and P_3. There are two types of orbits: those that appear as closed curves and those that connect both electrodes. The regions containing each type of trajectories are separated by a heteroclinic orbit connecting the pair of saddle points. This special orbit is of particular interest. As ions emitted at the injector cannot reach the region containing the closed curves, this region appears empty of charge. The heteroclinic, or separatrix, separates the charged and noncharged zones.

An analytical solution for this system was obtained in the case $A \sim 1$. Defining $x^* = x/L$, $z^* = z - 1/2$, $t^* = \pi^2 t$, $A^* = (A-1)/\pi^2$, and $H^* = H/\pi^2 L$, a Taylor expansion of H^* gives

$$H = x\left(A - \frac{x^2}{6} - z^2\right), \tag{6}$$

where asterisks have been omitted.

The behavior of the carrier trajectories when the velocity amplitude is perturbed with a periodic fluctuation $A = A_0 + \epsilon \sin \Omega t$ is then studied using Melnikov's method. In particular, Melnikov's function is computed to be

$$M(t_0) = -\epsilon \frac{\sqrt{6}\pi\Omega}{2} \frac{\sin \Omega t_0}{\cosh(\Omega\pi/4\sqrt{A})}. \tag{7}$$

Note that the parameter ϵ is now included in the definition of Melnikov's function. Since this function has an infinity of simple zeros, the main conclusion is that a chaotic layer connecting the injector with the otherwise charge-free region is always present. This chaotic layer produces a mixing of charge modifying the steady-state charge distribution.

In [6] it was conjectured that this chaotic behavior of the ion trajectories might be at the basis of the chaotic fluctuations of the electrical current that have been observed experimentally [5].

III. ROLE OF COULOMB REPULSION

When injection strength is weak but not negligible ($C < 1$) the Coulomb repulsion between ions has to be taken into account. This has two consequences. First, along the ions trajectories the charge density is no longer constant, but it decreases according to $q = q_0/(1 + q_0 t)$. On the other hand, the electric field is not constant, $\nabla \cdot \boldsymbol{E} = q$, and the system (4) is not conservative.

We consider Eqs. (4b) and (4c) with an electric field $\boldsymbol{E} = \boldsymbol{e}_z + \delta\boldsymbol{E}$, where $\nabla \cdot \delta\boldsymbol{E} = q$. Near $x = 0$, $z = 1/2$ we expand $\delta\boldsymbol{E}$ in a Taylor series: $\delta E_x = Cax$ and $\delta E_z = C(b + ez)$; this is consistent with the boundary conditions for the electric field: $E_x = 0$ at $x = 0$ and $\delta\Phi = 0$ at $z = 0, 1$.

The coefficients a and e are related to the charge in the inner region $q_{\text{inner}} = \nabla \cdot \delta\boldsymbol{E} = C(a+e)$. As the mechanism that introduces charge in this region is a laminar chaotic mixing, q_{inner} is a complicated function of $A(t)$. As a first approach we will consider a and e as constants. This is equivalent to considering an average charge in the inner region.

From

$$\frac{dx}{dt} = -2xz + Cax, \tag{8a}$$

$$\frac{dz}{dt} = -A - \epsilon \sin(\Omega t) + \frac{x^2}{2} + z^2 + C(b + ez), \tag{8b}$$

we compute Melnikov's function in the same way as it is done in [7,6,8]. We obtain

$$M(t_0) = \epsilon \frac{\sqrt{6}\pi\Omega}{2} \frac{\cos \Omega t_0}{\cosh(\Omega\pi/4\sqrt{A})} - \frac{\sqrt{6}\pi}{2} A q_{\text{inner}}. \tag{9}$$

From this equation it is clear that $M(t_0)$ has no simple zeros when q_{inner} is greater than a certain value. Hence Coulomb repulsion can break the heteroclinic tangle and suppress the chaotic behavior of the ion trajectories near the separatrix.

It is possible to obtain an upper bound to the charge in the inner region. As we have said, the mechanism that introduces charge in the inner region is that a nonsteady amplitude breaks the separatrix giving place to a chaotic layer that mixes charge between the inner and outer regions. When $M(t_0) = 0$ the stable and unstable manifolds corresponding to the two saddle points intersect transversely. These intersections define lobes in the manifolds. These lobes are sketched in Fig. 2. After a

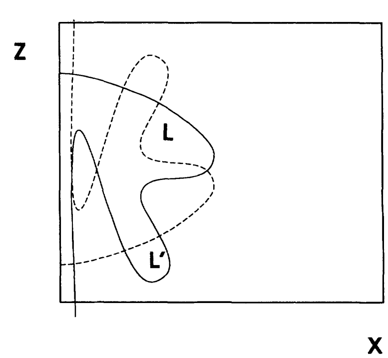

FIG. 2. Heteroclinic tangle. The lobe L maps into the lobe L' after one period of the amplitude oscillations.

period, the lobe L is mapped into the lobe L' introducing charge in the inner region. The succesive mapping of one lobe into others mixes the charge over the whole separatrix layer. A rough estimation of the charge in the inner layer is then

$$q_{\text{inner}} \simeq C\frac{S_l}{S_s}, \tag{10}$$

where S_l is the area of one lobe (as the mapping is area preserving when $C = 0$ this area is the same for all lobes) and S_s is the area of the unperturbed inner region. This will be an upper bound because the charge will decrease with time due to Coulomb repulsion [see Eq. (4a)].

The area of one lobe is given by

$$S_l = \int_{t_n}^{t_{n+1}} M(t)\,dt, \tag{11}$$

where t_n and t_{n+1} are two consecutive zeros of $M(t)$. When $C = 0$ we have

$$S_l = \epsilon\sqrt{6}\pi\Omega\frac{1}{\cosh(\Omega\pi/4\sqrt{A})}. \tag{12}$$

Also $S_s = \sqrt{6}A$ and substituting in (10) is

$$q_{\text{inner}} \simeq C\epsilon\pi\Omega\frac{1}{A\cosh(\Omega\pi/4\sqrt{A})}. \tag{13}$$

This is consistent with the numerical results in [6] where the inner region appears more charged for values of Ω at which the amplitude of Melnikov's function is a maximum [cf. [6], Eq. (27)].

Melnikov's function is finally

$$M(t_0) = \epsilon\frac{\sqrt{6}\pi\Omega}{2}\frac{1}{\cosh(\Omega\pi/4\sqrt{A})}[\sin(\Omega t_0) - 2C], \tag{14}$$

which has no zeros if $C > 0.5$.

IV. CONCLUSION

We have studied the charge-carrier trajectories when the liquid velocity is a periodic function of time and Coulomb repulsion is taken into account. It has been shown that for $C > 0.5$ Melnikov's function has no zeros. Therefore, for these values of C there is not heteroclinic chaos. From this we can conclude that the time-dependent behavior of the electrical current found experimentally for high values of C cannot be due to the chaotic dynamics of the ion trajectories near the separatrix.

For strong injection (high C) the chaotic behavior of the system may be due to the coupling between charge and velocity that appears because of the existence of a Coulomb force in the volume. This coupling may originate an energy interchange between modes analogous to that appearing in other fluid dynamics problems, e.g., the Rayleigh-Bénard problem. However, the presence of strong gradients in the spatial distribution of the charge density makes a modal expansion of the problem extremely difficult.

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